7-5 Magnetic Potentials

Reading Assignment: *pp. 227-236*

Recall that the definition of **electric scalar** potential:

$$\mathbf{E}(\bar{r}) = -\nabla V(\bar{r})$$

led to the **integral** relationship:

$$\int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = V(\bar{r}_a) - V(\bar{r}_b)$$

Q:

A:

HO: The Integral Definition of Magnetic Vector Potential

There are **numerous analogies** between electrostatics and magnetostatics!

Q:

A:

HO: The Magnetic Dipole

The Integral Definition of Magnetic Vector Potential

Recall for **electrostatics**, we began with the definition of **electric scalar potential**:

$$\mathbf{E}(\bar{r}) = -\nabla V(\bar{r})$$

And then taking a **contour** integral of each side we discovered:

$$\int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = -\int_C \nabla V(\bar{r}) \cdot \overline{d\ell}$$

$$\int_C \mathbf{E}(\bar{r}) \cdot \overline{d\ell} = V(\bar{r}_a) - V(\bar{r}_b)$$

We can perform an **analogous** procedure for magnetic vector potential! Recall magnetic flux density $\mathbf{B}(\bar{r})$ can be written in terms of the magnetic vector potential $\mathbf{A}(\bar{r})$:

$$\mathbf{B}(\bar{r}) = \nabla \times \mathbf{A}(\bar{r})$$

Say we **integrate** both sides over some **surface** S:

$$\iint_S \mathbf{B}(\bar{r}) \cdot \overline{ds} = \iint_S \nabla \times \mathbf{A}(\bar{r}) \cdot \overline{ds}$$

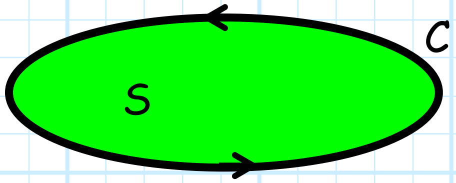
We can apply **Stoke's theorem** to write the right side as:

$$\iint_S \nabla \times \mathbf{A}(\bar{r}) \cdot \overline{ds} = \oint_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell}$$

Therefore, we find that we can also define magnetic vector potential in an **integral form** as:

$$\iint_S \mathbf{B}(\bar{r}) \cdot \overline{ds} = \oint_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell}$$

where contour C defines the **border** of surface S.



Consider now the **meaning** of the integral:

$$\iint_S \mathbf{B}(\bar{r}) \cdot \overline{ds}$$

This integral is remarkably **similar** to:

$$\iint_S \mathbf{J}(\bar{r}) \cdot \overline{ds}$$

where:

$$\mathbf{B}(\bar{r}) \doteq \text{magnetic flux } \textbf{density} \quad \left[\frac{Webers}{meters^2}\right]$$

and:

$$\mathbf{J}(\bar{r}) \doteq \text{current } \textbf{density} \quad \left[\frac{Amperes}{meters^2}\right]$$

Recall that integrating the **current density** (in *amps/m²*) over some surface S (in m^2), provided us the **total current** I flowing through surface S:

$$\iint_S \mathbf{J}(\bar{r}) \cdot \overline{ds} = I$$

Similarly, integrating the **magnetic flux density** (in *webers/m²*) over some surface S (in m^2), provided us the **total magnetic flux** Φ flowing through surface S:

$$\iint_S \mathbf{B}(\bar{r}) \cdot \overline{ds} = \Phi$$

where Φ is defined as:

$$\Phi \doteq \text{magnetic flux} \quad [Webers]$$

Using the equations derived previously, we can **directly** relate magnetic vector potential $\mathbf{A}(\bar{r})$ to magnetic flux as:

$$\Phi = \oint_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell}$$

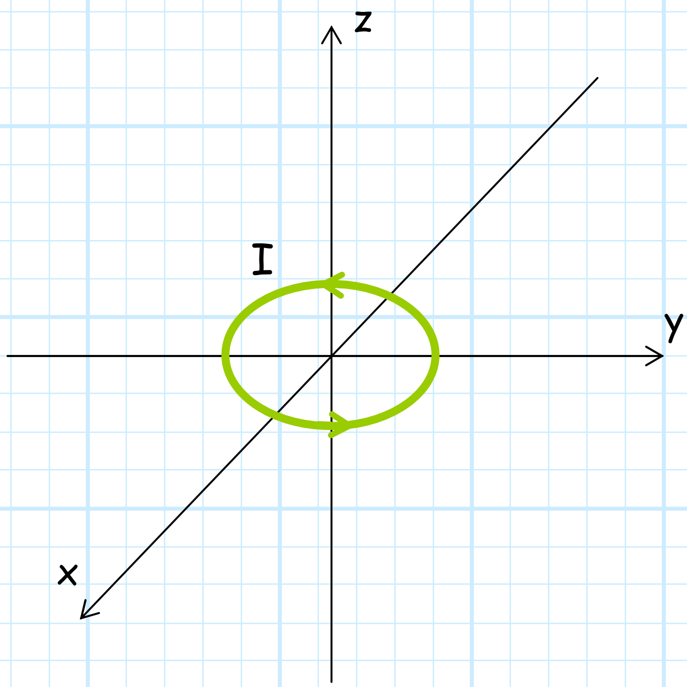
where we recall that the **units** for magnetic vector potential are *Webers/m.*

Note the similarities of the above expression to the integral form of **Ampere's Law!**

$$I = \frac{1}{\mu_0} \oint_C \mathbf{B}(\bar{r}) \cdot \overline{d\ell}$$

The Magnetic Dipole

Consider a **very small,** circular **current loop** of radius *a*, carrying current *I*.



Q: What **magnetic flux density** is produced by this loop, in regions far from the current (i.e., *r*>>*a*)?

A: First, find the magnetic vector potential $\mathbf{A}(\bar{r})$:

$$\mathbf{A}(\bar{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{I\, \overline{d\ell'}}{|\bar{r} - \bar{r}'|}$$

Since the contour *C* is a **circle** around the *z*-axis, with radius *a*, we use the **differential line vector**:

$$\begin{aligned} \overline{d\ell'} &= \rho' d\phi' \, \hat{a}_\phi \\ &= a \, d\phi' \, \hat{a}_\phi \\ &= \left(a \cos\phi' \, \hat{a}_x + a \sin\phi' \, \hat{a}_y \right) d\phi' \end{aligned}$$

The location of the current is specified by position vector $\bar{r}'$. Since for **every point** on the current loop we find $z' = 0$ and $\rho' = a$, we find:

$$\begin{aligned}\bar{r}' &= x'\,\hat{a}_x + y'\,\hat{a}_y + z'\,\hat{a}_z \\ &= \rho' \cos\phi'\,\hat{a}_x + \rho' \sin\phi'\,\hat{a}_y + z'\,\hat{a}_z \\ &= a\cos\phi'\,\hat{a}_x + a\sin\phi'\,\hat{a}_y\end{aligned}$$

And finally,

$$\begin{aligned}\bar{r} &= x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z \\ &= r\sin\theta\cos\phi\,\hat{a}_x + r\sin\theta\sin\phi\,\hat{a}_y + r\cos\theta\,\hat{a}_z\end{aligned}$$

With a little **algebra** and **trigonometry**, we find also that:

$$\frac{1}{|\bar{r}-\bar{r}'|} = \left[r^2 - a\left(2r\sin\theta\cos(\phi-\phi')\right) + a^2\right]^{-1/2}$$

Since the **radius** of the circle is **very small** (i.e., $a \ll r$), we can use a Taylor Series to approximate the above expression (see page 231 of text):

$$\frac{1}{|\bar{r}-\bar{r}'|} \approx \frac{1}{r} + \frac{a\sin\theta\cos(\phi-\phi')}{r^2}$$

The magnetic vector potential can now be **evaluated**!

$$\begin{aligned}\mathbf{A}(\bar{r}) &= \frac{\mu_0}{4\pi}\oint_C \frac{I\,\overline{d\ell'}}{|\bar{r}-\bar{r}'|} \\ &= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\left(\frac{1}{r}+\frac{a\sin\theta\cos(\phi-\phi')}{r^2}\right)\left(a\cos\phi'\,\hat{a}_x + a\sin\phi'\,\hat{a}_y\right)d\phi' \\ &= \frac{\pi a^2 I}{r^2}\sin\theta\left(-\sin\phi\,\hat{a}_x+\cos\phi\,\hat{a}_y\right) \\ &= \frac{\pi a^2 I}{r^2}\sin\theta\,\hat{a}_\phi\end{aligned}$$

Note that πa^2 equals the **surface area** S of the circular loop. Therefore, we can write that magnetic vector potential produced by a very small current loop is:

$$\mathbf{A}(\bar{r}) = \frac{\mu_0 S I}{4\pi r^2}\sin\theta\,\hat{a}_\phi \qquad (a \ll r)$$

We can **now** determine magnetic flux density $\mathbf{B}(\bar{r})$ by taking the **curl**:

$$\begin{aligned}\mathbf{B}(\bar{r}) &= \nabla \mathbf{x} \mathbf{A}(\bar{r}) \\ &= \frac{\mu_0 S I}{4\pi r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)\end{aligned}$$

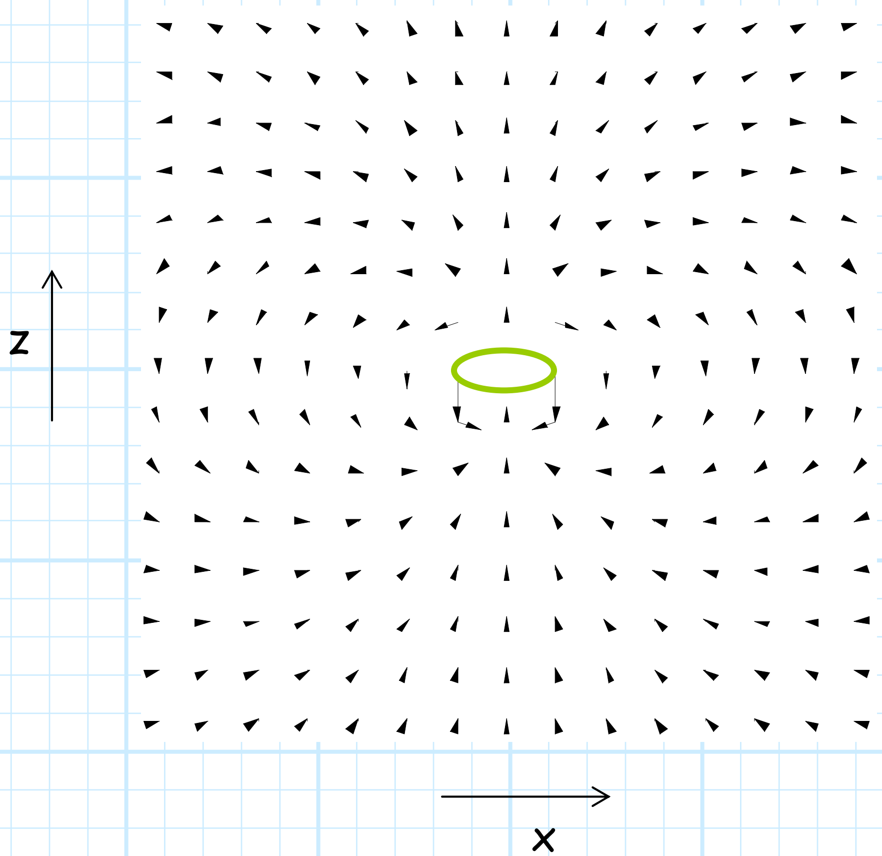
Q: *Hey! Something about this result looks very familiar !*

A: Compare this result to that of an **electric dipole**:

$$\mathbf{E}(\bar{r}) = \frac{Qd}{4\pi\varepsilon r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)$$

Both results have exactly the **same** form!:

$$c\left(\frac{2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta}{4\pi r^3}\right)$$



where c is a constant.

Because of this similarity, we can refer to a small current loop of are S and current I as a **Magnetic Dipole**.

Note that the only difference between the mathematical description of an **electric** field produced by an **electric** dipole and the **magnetic** flux density produced by a **magnetic** dipole is a **constant** c:

$$\text{electric dipole} \quad \rightarrow \quad c = \frac{Qd}{\varepsilon}$$

$$\text{magnetic dipole} \quad \rightarrow \quad c = \mu_0 S\, I$$

Recall that we defined a **dipole moment** for electric dipoles, where:

$$|\mathbf{p}| = Qd$$

Clearly, the **analogous** product to Qd for a magnetic dipole is SI. We can, in fact, define a **magnetic dipole moment m**:

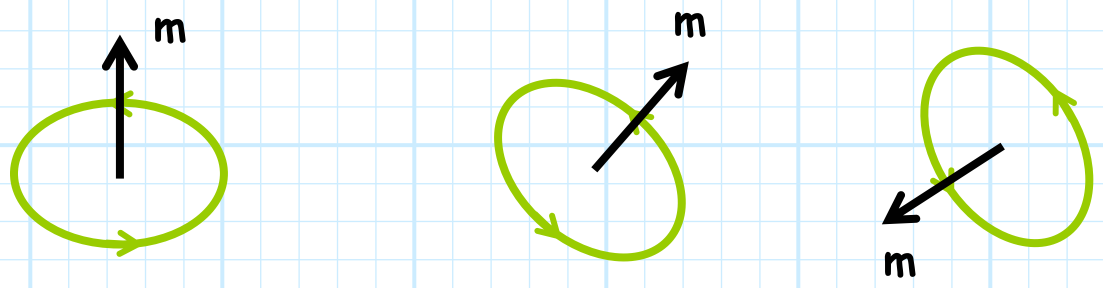
$$\mathbf{m} \doteq \text{Magnetic Dipole Moment} \qquad \left[\text{Amps} \cdot \text{m}^2\right]$$

Analogous to the electric dipole, the magnetic dipole moment has **magnitude**:

$$|\mathbf{m}| = SI$$

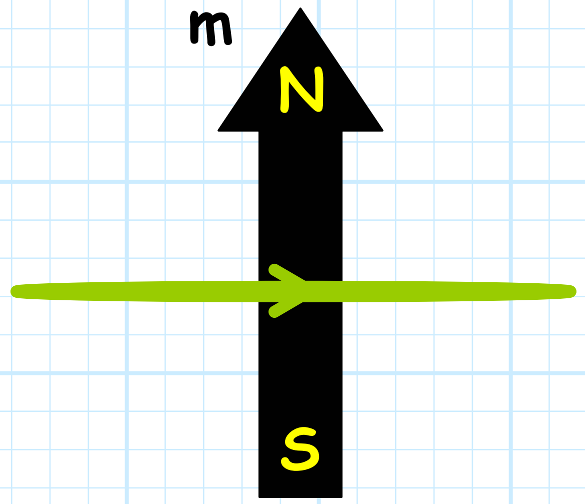
Q: *We now know the magnitude of the magnetic dipole moment, but what is its **direction** ??*

A: The magnetic dipole **m** points in the direction **orthogonal** to the circular surface *S*, e.g.:



Note the direction is defined using the **right-hand rule** with respect to the direction of current *I*.

Instead of plus (+) and minus (-), the **poles** of a magnetic dipole are defined as **north** (N) and **south**(S):



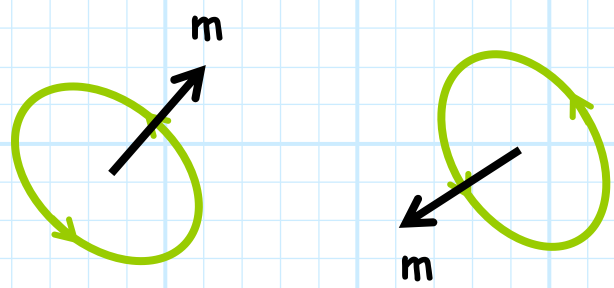
Thus, for the **example** provide on this handout, the magnetic dipole moment is:

$$\mathbf{m} = S I \, \hat{a}_z$$

We note that $S I \sin\theta \, \hat{a}_\phi = S I \, \hat{a}_z \times \hat{a}_r = \mathbf{m} \times \hat{a}_r$, therefore we can write:

$$\begin{aligned} \mathbf{A}(\bar{r}) &= \frac{\mu_0 \, S \, I}{4\pi r^2} \sin\theta \, \hat{a}_\phi \\ &= \frac{\mu_0 \, \mathbf{m} \times \hat{a}_r}{4\pi r^2} \end{aligned}$$

The above equation is in fact valid for **any** magnetic dipole **m** located at the origin, **regardless** of its direction! In other words, we can also use the above expression if **m** is pointed in some direction **other** than $\hat{a}_z$, e.g.:



Q: What if the magnetic dipole is **not** located at the **origin**?

A: Just like we have **many** times before, we make the substitutions:

$$r \rightarrow |\bar{r} - \bar{r}'| \qquad\qquad \hat{a}_r = \hat{a}_R = \frac{\bar{r} - \bar{r}'}{|\bar{r} - \bar{r}'|}$$

Therefore, we find the magnetic flux density $\mathbf{A}(\bar{r})$ produced by an **arbitrary** magnetic dipole **m**, located at an **arbitrary** position $\bar{r}'$, is:

$$\mathbf{A}(\bar{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times (\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3}$$

To determine the magnetic flux density $\mathbf{B}(\bar{r})$, we simply take the **curl** of the above expression.

Note this is analogous to the expression of the **electric scalar potential** generated by an **electric dipole** with moment **p**:

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon} \frac{\mathbf{p} \cdot (\bar{r} - \bar{r}')}{|\bar{r} - \bar{r}'|^3}$$

and then taking the gradient of this function to determine the **electric field** $\mathbf{E}(\bar{r})$.